EQUATIONS OF THE LINEAR THEORY OF ELASTICITY

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The present study introduces eigenvalues and eigenvectors for the coefficient matrices of the equations of motion, in displacements, of the linear theory of elasticity. The eigenvalues and eigenvectors are found for materials having a crystalline structure and are represented through independent parameters which ensure that the specific strain energy will be positive-definite. The materials for which the equations for each displacement are independent are indicated later in the text. The equations of motion are divided into 32 classes, depending on the number of different eigenvalues and their multiplicity.

In orthogonal Cartesian coordinates x_1, x_2, x_3, the equations of the theory of elasticity have the form

$$\left[\frac{1}{2}(A_{iklj}+A_{ilkj})\partial_{kl}-\rho\delta_{ij}\partial_{..}\right]u_j+F_i=0, \tag{1}$$

where u_j is the displacement vector; F_i is the vector of the body forces; $A_{ik\ell j}$ is the tensor of the elastic moduli; ρ is the constant density of the material; δ_{ij} is the Kronecker symbol; ∂_k denotes differentiation with respect to the coordinate x_k; $\partial.$ denotes differentiation with respect to time; repeating alphabetical subscripts denote summation from 1 to 3. The constants $A_{ik\ell j}$ have properties of symmetry [1]:

$$A_{iklj}=A_{kilj}=A_{ljik}. \tag{2}$$

Taking (2) into account, we find that the coefficients $A^*_{ijk\ell}=A_{i(k\ell)j}=1/2\cdot(A_{ik\ell j}+A_{i\ell kj})$ in (1) have the same symmetry properties as the elastic moduli:

$$A^*_{ijkl}=A^*_{jikl}=A^*_{klij}. \tag{3}$$

In a manner analogous to the characteristic elastic moduli and eigenstates [2-8] for the tensor $A_{ik\ell j}$, we can introduce eigenvalues and characteristic tensors for coefficients (3):

$$A^*_{ijkl}=f_{ijpq}\mu_{pqrs}f_{klrs}\quad((pq)=(rs)), \tag{4}$$

$$f_{ijpq}f_{ijrs}=\delta_{pqrs}=\frac{1}{2}(\delta_{pr}\delta_{qs}+\delta_{ps}\delta_{qr}).$$

Equations (4) appear as follows in the double-index matrix notation in [6-8]:

$$A^*_{ik}=f_{ip}\mu_{pr}f_{kr}\quad(p=r), \tag{5}$$

$$f_{ip}f_{ir}=\delta_{pr}.$$

Here, summation is performed over repeating indices from 1 to 6. It is obvious that $\mu_{11}=\mu_1$, $\mu_{22}=\mu_2$, ..., μ_6, with f_{ip} representing eigenvalues and eigenvectors of the symmetric matrix A^*_{ik}.

The matrices A^*_{ik} and A_{ij} have a one-to-one correspondence which can be expressed through the formulas

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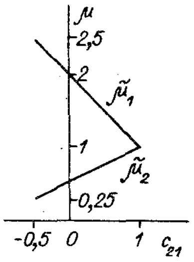

Fig. 1

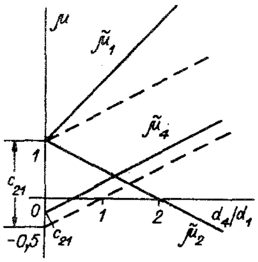

Fig. 2

$$A^*_{ik} = \begin{bmatrix} A_{11} & & & & & \\ \frac{1}{2}A_{66} & A_{22} & & & \text{sym} & \\ \frac{1}{2}A_{55} & \frac{1}{2}A_{44} & A_{33} & & & \\ \frac{1}{\sqrt{2}}A_{65} & A_{42} & A_{43} & \frac{1}{2}A_{44}+A_{32} & & \\ A_{51} & \frac{1}{\sqrt{2}}A_{64} & A_{53} & \frac{1}{2}A_{54}+\frac{1}{\sqrt{2}}A_{63} & \frac{1}{2}A_{55}+A_{31} & \\ A_{61} & A_{62} & \frac{1}{\sqrt{2}}A_{54} & \frac{1}{2}A_{64}+\frac{1}{\sqrt{2}}A_{52} & \frac{1}{2}A_{65}+\frac{1}{\sqrt{2}}A_{41} & \frac{1}{2}A_{66}+A_{21} \end{bmatrix}; \qquad (6)$$

$$A_{ij} = \begin{bmatrix} A^*_{11} & & & & & \\ A^*_{66}-A^*_{21} & A^*_{22} & & & \text{sym} & \\ A^*_{55}-A^*_{31} & A^*_{44}-A^*_{32} & A^*_{33} & & & \\ \sqrt{2}A^*_{65}-A^*_{41} & A^*_{42} & A^*_{43} & 2A^*_{32} & & \\ A^*_{51} & \sqrt{2}A^*_{64}-A^*_{52} & A^*_{53} & \sqrt{2}A^*_{63} & 2A^*_{31} & \\ A^*_{61} & A^*_{62} & \sqrt{2}A^*_{54}-A^*_{63} & \sqrt{2}A^*_{52} & \sqrt{2}A^*_{41} & 2A^*_{21} \end{bmatrix}. \qquad (7)$$

Thus, elastic materials can be specified by means of Eq. (7) if we know A^*_{ik}. However, here it is necessary to ensure satisfaction of conditions of positive-definiteness [9] for A_{ij}. In [9], A_{ij} was represented through the independent parameters d_i and c_{ip} in the form

$$\begin{gathered} A_{ij} = d_1c_{i1}c_{j1} + d_2c_{i2}c_{j2} + d_3c_{i3}c_{j3} + d_4c_{i4}c_{j4} + \\ + d_5c_{i5}c_{j5} + d_6c_{i6}c_{j6},\ c_{ip} = 0\ (p > i),\ c_{11} = \ldots = c_{66} = 1. \end{gathered} \qquad (8)$$

For matrix A_{ij} of (8) to be positive-definite, it is necessary and sufficient that the conditions $d_i > 0$, $i = 1, \ldots, 6$ be satisfied [9]. The parameters c_{ip} $(i > p)$ can take any real values. In general form, the matrix A^*_{ik} is not positive-definite. For materials satisfying the Cauchy relations [10] $A_{i[k\ell]j} = 1/2\cdot(A_{ik\ell j} - A_{i\ell kj}) = 0$, matrices (6) and (7) coincide.

Characteristic elastic moduli $\lambda_i > 0$ were presented in [7] along with eigenstates t_{ip} for materials with crystallographic syngony. Using the formulas from [7] and considering Eqs. (8), we find the eigenvalues μ_i and eigenvectors f_{ip} of matrix (6) for such materials.

Isotropic material

$$\begin{gathered} \mu_1 = 2A_{11} - A_{21} = d_1(2 - c_{21}),\ -\frac{1}{2} < c_{21} < 1,\ \mu_2 = \mu_3 = \ldots = \mu_6 = \\ = \frac{1}{2}(A_{11} + A_{21}) = \frac{1}{2}d_1(1 + c_{21}); \end{gathered}$$

$$f_{ip}=\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \quad (9)$$

The graphs of $\tilde{\mu}_i = \mu_i/d_1$ are shown in Fig. 1. It is evident that $1 < \tilde{\mu}_1 < 2.5$ and $0.25 < \tilde{\mu}_2 < 1$.

Cubic syngony

$$\mu_1 = A_{11} + A_{44} = d_1(1 + d_4/d_1),\ \ \mu_2 = \mu_3 = A_{11} - \frac{1}{2}A_{44} = d_1\left(1 - \frac{1}{2}d_4/d_1\right),$$
$$\mu_4 = \mu_5 = \mu_6 = \frac{1}{2}A_{44} + A_{21} = d_1\left(\frac{1}{2}d_4/d_1 + c_{21}\right),\ \ -\frac{1}{2} < c_{21} < 1.$$

The eigenvectors f_{ip} are given by matrix (9). The graphs of $\tilde{\mu}_i$ are shown in Fig. 2. It is evident that $\tilde{\mu}_2$ and $\tilde{\mu}_4$ can take negative and zero values. The graph of $\tilde{\mu}_4$ passes parallel to the dashed lines at any point within the band, depending on the values of c_{21}. If $\mu_4 = 0$, then for each displacement u_i Eqs. (1) become independent of one another:

$$\left[A_{11}\partial_{11} + \frac{1}{2}A_{44}(\partial_{22} + \partial_{33}) - \rho\partial_{..}\right]u_1 + F_1 = 0,$$
$$\left[\frac{1}{2}A_{44}\partial_{11} + A_{11}\partial_{22} + \frac{1}{2}A_{44}\partial_{33} - \rho\partial_{..}\right]u_2 + F_2 = 0,$$
$$\left[\frac{1}{2}A_{44}(\partial_{11} + \partial_{22}) + A_{11}\partial_{33} - \rho\partial_{..}\right]u_3 + F_3 = 0.$$

The matrix A_{ij} of Hooke's law for this case has the form

$$A_{ij}=\begin{bmatrix} d_1 & & & & & \\ -\frac{1}{2}d_4 & d_1 & & \text{sym} & & \\ -\frac{1}{2}d_4 & -\frac{1}{2}d_4 & d_1 & & & \\ 0 & 0 & 0 & d_4 & & \\ 0 & 0 & 0 & 0 & d_4 & \\ 0 & 0 & 0 & 0 & 0 & d_4 \end{bmatrix},\quad 0 < d_4 < d_1.$$

These are materials with a negative Poisson's ratio: $-1 < \nu = -A^{-1}_{21}/A^{-1}_{11} = -d_4/(2d_1 - d_4) < 0$. Here, A^{-1}_{ij} are elements of the inverse matrix. An example of a material with cubic syngony and a negative Poisson's ratio is pyrite [10, p. 175].

Hexagonal syngony (transverse isotropy)

$$\mu_{1,2} = \frac{1}{2}\left[\frac{1}{2}(3A_{11} - A_{21}) + A_{33} \pm \sqrt{\left(\frac{1}{2}(3A_{11} - A_{21}) - A_{33}\right)^2 + 2A_{44}^2}\right] =$$
$$= \frac{1}{2}d_1\left[\frac{1}{2}(3 - c_{21}) + \frac{2c_{31}^2}{1+c_{21}} + \frac{d_3}{d_1} \pm \sqrt{\left(\frac{1}{2}(3 - c_{21}) - \frac{2c_{31}^2}{1+c_{21}} - \frac{d_3}{d_1}\right)^2 + 2\left(\frac{d_4}{d_1}\right)^2}\right];$$

$$\mu_3=\mu_6=\frac{1}{2}(A_{11}+A_{21})=\frac{1}{2}d_1(1+c_{21}),\ -1<c_{21}<1,$$
$$\mu_4=\mu_5=\frac{1}{2}A_{44}+A_{31}=d_1\left(\frac{1}{2}d_4/d_1+c_{31}\right); \tag{10}$$

$$f_{ip}=\begin{bmatrix} \frac{1}{\sqrt{2}}\cos\alpha & \frac{-1}{\sqrt{2}}\sin\alpha & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}}\cos\alpha & \frac{-1}{\sqrt{2}}\sin\alpha & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}; \tag{11}$$

$$\operatorname{tg}2\alpha=\frac{\sqrt{2}A_{44}}{\frac{1}{2}(3A_{11}-A_{21})-A_{33}}=\frac{\sqrt{2}\,d_4/d_1}{\frac{1}{2}(3-c_{21})-\frac{2c_{31}^2}{1+c_{21}}-\frac{d_3}{d_1}}. \tag{12}$$

It follows from (10) that $\mu_1 > 0$, so that

$$\mu_2\geqslant 0,\quad \text{if}\quad (3-c_{21})\left(\frac{2c_{31}^2}{1+c_{21}}+\frac{d_3}{d_1}\right)\geqslant\left(\frac{d_4}{d_1}\right)^2,$$
$$\mu_2<0,\quad \text{if}\quad (3-c_{21})\left(\frac{2c_{31}^2}{1+c_{21}}+\frac{d_3}{d_1}\right)<\left(\frac{d_4}{d_1}\right)^2. \tag{13}$$

The graphs of $\tilde{\mu}_3$ and $\tilde{\mu}_4$ are shown in Figs. 3 and 4. It is evident that $0 < \tilde{\mu}_3 < 1$. The graph of $\tilde{\mu}_4$ passes parallel to and above the dashed line at an arbitrary distance which depends on the values of the parameter $(1/2)d_4/d_1$. If $\mu_4 = 0$, then we can use (1) to obtain a separate equation for the displacement u_3:

$$\left[A_{11}\partial_{11}+\frac{1}{2}(A_{11}-A_{21})\partial_{22}+\frac{1}{2}A_{44}\partial_{33}-\rho\partial_{..}\right]u_1+\frac{1}{2}(A_{11}+A_{21})\partial_{12}u_2+F_1=0,$$
$$\frac{1}{2}(A_{11}+A_{21})\partial_{21}u_1+\left[\frac{1}{2}(A_{11}-A_{21})\partial_{11}+A_{11}\partial_{22}+\frac{1}{2}A_{44}\partial_{33}-\rho\partial_{..}\right]u_2+F_2=0,$$
$$\left[\frac{1}{2}A_{44}(\partial_{11}+\partial_{22})+A_{33}\partial_{33}-\rho\partial_{..}\right]u_3+F_3=0.$$

In this case, the matrix A_{ij} has the form

$$A_{ij}=\begin{bmatrix} d_1 & & & & & \\ d_1c_{21} & d_1 & & \text{sym} & & \\ -\frac{1}{2}d_4 & -\frac{1}{2}d_4 & \frac{d_4^2}{2d_1(1+c_{21})}+d_3 & & & \\ 0 & 0 & 0 & d_4 & & \\ 0 & 0 & 0 & 0 & d_4 & \\ 0 & 0 & 0 & 0 & 0 & d_1(1-c_{21}) \end{bmatrix}.$$

The Poisson's ratios for such a material are:

$$-1<\nu_{21}=\nu_{12}=-\frac{A_{21}^{-1}}{A_{11}^{-1}}=\frac{4d_1d_3c_{21}(1+c_{21})-d_4^2(1-c_{21})}{4d_1d_3(1+c_{21})+d_4^2(1-c_{21})}<1,$$
$$\nu_{31}=\nu_{32}=-\frac{A_{31}^{-1}}{A_{11}^{-1}}=\frac{-2d_1d_4(1-c_{21}^2)}{4d_1d_3(1+c_{21})+d_4^2(1-c_{21})}<0,$$
$$\nu_{13}=\nu_{23}=-\frac{A_{31}^{-1}}{A_{33}^{-1}}=\frac{-d_4}{2d_1(1+c_{21})}<0.$$

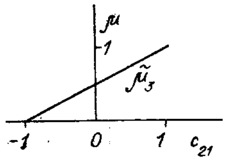

Fig. 3

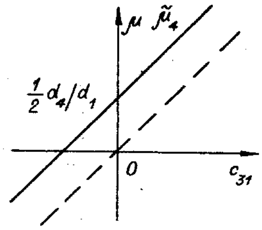

Fig. 4

Trigonal syngony

$$\mu_{3,4}=\frac{1}{2}\left[\frac{1}{2}(A_{11}+A_{21}+A_{44})+A_{31}\pm\sqrt{\left(\frac{1}{2}(A_{11}+A_{21}-A_{44})-A_{31}\right)^2+8A_{41}^2}\right]=$$

$$=\frac{1}{2}d_1\left[\frac{1}{2}\left(1+c_{21}+\frac{2c_{41}^2}{1-c_{21}}+\frac{d_4}{d_1}\right)+c_{31}\pm\right.$$

$$\left.\pm\sqrt{\left(\frac{1}{2}\left(1+c_{21}-\frac{2c_{41}^2}{1-c_{21}}-\frac{d_4}{d_1}\right)-c_{31}\right)^2+8c_{41}^2}\right],$$

$$\mu_5=\mu_4,\quad \mu_6=\mu_3,\quad -1<c_{21}<1,$$

$$f_{ip}=\begin{bmatrix} \frac{1}{\sqrt{2}}\cos\alpha & \frac{-1}{\sqrt{2}}\sin\alpha & \frac{1}{\sqrt{2}}\cos\beta & \frac{-1}{\sqrt{2}}\sin\beta & 0 & 0 \\ \frac{1}{\sqrt{2}}\cos\alpha & \frac{-1}{\sqrt{2}}\sin\alpha & \frac{-1}{\sqrt{2}}\cos\beta & \frac{1}{\sqrt{2}}\sin\beta & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & \sin\beta & \cos\beta & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\beta & \sin\beta \\ 0 & 0 & 0 & 0 & -\sin\beta & \cos\beta \end{bmatrix},$$

$$\mathrm{tg}\,2\beta=\frac{2\sqrt{2}A_{41}}{\frac{1}{2}(A_{11}+A_{21}-A_{44})-A_{31}}=\frac{2\sqrt{2}c_{41}}{\frac{1}{2}\left(1+c_{21}-\frac{2c_{41}^2}{1-c_{21}}-\frac{d_4}{d_1}\right)-c_{31}}.$$

The eigenvalues μ_1 and μ_2 and angle α are given by Eqs. (10) and (12). We also have the inequalities (13) $\mu_1 > 0$, $\mu_3 > 0$ with $\mu_4 \geq 0$, if $d_4/2d_1 + c_{31} \geq c_{41}^2(4/(1 + c_{21}) - 1/(c - c_{21}))$, $\mu_4 < 0$, if $d_4/2d_1 + c_{31} < c_{41}^2(4/(1 + c_{21}) - 1/(1 - c_{21}))$.

Tetragonal syngony

$$\mu_{1,2}=\frac{1}{2}\left[A_{11}+\frac{1}{2}A_{66}+A_{33}\pm\sqrt{\left(A_{11}+\frac{1}{2}A_{66}-A_{33}\right)^2+2A_{44}^2}\right]=$$

$$=\frac{1}{2}d_1\left[1+\frac{d_6}{2d_1}+\frac{2c_{31}^2}{1+c_{21}}+\frac{d_3}{d_1}\pm\sqrt{\left(1+\frac{d_6}{2d_1}-\frac{2c_{31}^2}{1+c_{21}}-\frac{d_3}{d_1}\right)^2+2\left(\frac{d_4}{d_1}\right)^2}\right]; \qquad (14)$$

$$\mu_3=A_{11}-\frac{1}{2}A_{66}=d_1\left(1-\frac{1}{2}d_6/d_1\right),$$

$$\mu_4=\mu_5=\frac{1}{2}A_{44}+A_{31}=d_1\left(\frac{1}{2}d_4/d_1+c_{31}\right),$$

$$\mu_6=\frac{1}{2}A_{66}+A_{21}=d_1\left(\frac{1}{2}d_6/d_1+c_{21}\right),\quad -1<c_{21}<1.$$

The eigenvectors f_{ip} are given by matrix (11). Here

$$\mathrm{tg}\,2\alpha=\frac{\sqrt{2}A_{44}}{A_{11}+\frac{1}{2}A_{66}-A_{33}}=\frac{\sqrt{2}\,d_4/d_1}{1+\frac{d_6}{2d_1}-\frac{2c_{31}^2}{1+c_{21}}-\frac{d_3}{d_1}}.$$

The graph of $\tilde{\mu}_4$ is shown in Fig. 4, while the graphs of $\tilde{\mu}_3$, $\tilde{\mu}_6$ are shown in Fig. 5. It is evident that $\tilde{\mu}_3$, $\tilde{\mu}_4$, $\tilde{\mu}_6$ can take negative and zero values. The graph of $\tilde{\mu}_6$ passes parallel

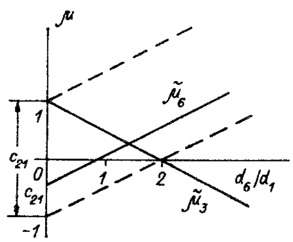

Fig. 5

to the dashed lines at any point within the band, depending on the values of the parameter c_{21}. It follows from (14) that $\mu_1 > 0$ and $\mu_2 \geq 0$ if $(1 + d_6/2d_1)(2c_{31}^2/(1 + c_{21}) + d_3/d_1) \geq 1/2(d_4/d_1)^2$, while $\mu_2 < 0$ if $(1 + d_6/2d_1)(2c_{31}^2/(1 + c_{21}) + d_3/d_1) < 1/2(d_4/d_1)^2$. If $\mu_4 = 0$ and $\mu_6 = 0$, then for each displacement u_i Eqs. (1) become independent of one another:

$$\left(A_{11}\partial_{11} + \frac{1}{2}A_{66}\partial_{22} + \frac{1}{2}A_{44}\partial_{33} - \rho\partial_{..}\right)u_1 + F_1 = 0,$$

$$\left(\frac{1}{2}A_{66}\partial_{11} + A_{11}\partial_{22} + \frac{1}{2}A_{44}\partial_{33} - \rho\partial_{..}\right)u_2 + F_2 = 0,$$

$$\left[\frac{1}{2}A_{44}(\partial_{11} + \partial_{22}) + A_{33}\partial_{33} - \rho\partial_{..}\right]u_3 + F_3 = 0.$$

The matrix A_{ij} of Hooke's law has the following form in this case

$$A_{ij} = \begin{bmatrix} d_1 & & & & & \\ -\frac{1}{2}d_6 & d_1 & & \text{sym} & & \\ -\frac{1}{2}d_4 & -\frac{1}{2}d_4 & \frac{d_4^2}{2d_1 - d_6} + d_3 & & & \\ 0 & 0 & 0 & d_4 & & \\ 0 & 0 & 0 & 0 & d_4 & \\ 0 & 0 & 0 & 0 & 0 & d_6 \end{bmatrix}, \quad 0 < d_6 < 2d_1.$$

The Poisson's ratios for such a material are as follows:

$$1 < \nu_{21} = \nu_{12} = -\frac{A_{21}^{-1}}{A_{11}^{-1}} = -\frac{2d_3d_6(2d_1 - d_6) + d_4^2(2d_1 + d_6)}{4d_1d_3(2d_1 - d_6) + d_4^2(2d_1 + d_6)} < 0,$$

$$\nu_{31} = \nu_{32} = -\frac{A_{31}^{-1}}{A_{11}^{-1}} = -\frac{d_4(4d_1^2 - d_6^2)}{4d_1d_3(2d_1 - d_6) + d_4^2(2d_1 + d_6)} < 0,$$

$$\nu_{13} = \nu_{23} = -\frac{A_{31}^{-1}}{A_{33}^{-1}} = \frac{-d_4}{2d_1 - d_6} < 0.$$

Rhombic syngony (orthotropic)

$$\mu_4 = \frac{1}{2}A_{44} + A_{32} = d_1\left(\frac{1}{2}d_4/d_1 + c_{31}c_{21} + c_{32}d_2/d_1\right),$$

$$\mu_5 = \frac{1}{2}A_{55} + A_{31} = d_1\left(\frac{1}{2}d_5/d_1 + c_{31}\right), \quad \mu_6 = \frac{1}{2}A_{66} + A_{21} = d_1\left(\frac{1}{2}d_6/d_1 + c_{21}\right),$$

$$f_{ip} = \begin{bmatrix} f_{11} & f_{12} & f_{13} & 0 & 0 & 0 \\ f_{21} & f_{22} & f_{23} & 0 & 0 & 0 \\ f_{31} & f_{32} & f_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad f_{ip}f_{iq} = \delta_{pq}.$$

The eigenvalues μ_1, μ_2, and μ_3 are roots of the third-degree equation

$$\begin{vmatrix} d_1-\mu & & \\ \frac{1}{2}d_6 & d_1c_{21}^2+d_2-\mu & \text{sym} \\ \frac{1}{2}d_5 & \frac{1}{2}d_4 & d_1c_{31}^2+d_2c_{32}^2+d_3-\mu \end{vmatrix}=0$$

and depend on the parameters $d_i > 0$, c_{ip}. Meanwhile $\mu_1>0$, $|\mu_2|<\mu_1$, $|\mu_3|<\mu_1$ [11]. The graphs of $\tilde{\mu}_4$, $\tilde{\mu}_5$, $\tilde{\mu}_6$ are similar to those shown in Fig. 4. If $\mu_4 = 0$, $\mu_5 = 0$, $\mu_6 = 0$, then for each displacement u_i Eqs. (1) become independent of one another:

$$\left(A_{11}\partial_{11}+\frac{1}{2}A_{66}\partial_{22}+\frac{1}{2}A_{55}\partial_{33}-\rho\partial_{..}\right)u_1+F_1=0,$$
$$\left(\frac{1}{2}A_{66}\partial_{11}+A_{22}\partial_{22}+\frac{1}{2}A_{44}\partial_{33}-\rho\partial_{..}\right)u_2+F_2=0,$$
$$\left(\frac{1}{2}A_{55}\partial_{11}+\frac{1}{2}A_{44}\partial_{22}+A_{33}\partial_{33}-\rho\partial_{..}\right)u_3+F_3=0.$$

In this case, we write the matrix A_{ij} of Hooke's law as

$$A_{ij}=\begin{bmatrix} d_1 & & & & & \\ -\frac{1}{2}d_6 & \frac{d_6^2}{4d_1}+d_2 & & \text{sym} & & \\ -\frac{1}{2}d_5 & -\frac{1}{2}d_4 & \frac{d_5^2}{4d_1}+\frac{d_4^2}{4d_2}\left(1+\frac{d_5d_6}{2d_1d_4}\right)+d_3 & & & \\ 0 & 0 & 0 & d_4 & & \\ 0 & 0 & 0 & 0 & d_5 & \\ 0 & 0 & 0 & 0 & 0 & d_6 \end{bmatrix}.$$

The Poisson's coefficients for such a material have the form:

$$\nu_{23}=-\frac{A_{23}^{-1}}{A_{33}^{-1}}=-\frac{d_4}{2d_2}\left(1+\frac{d_5d_6}{2d_1d_4}\right)<0,\quad \nu_{13}=-\frac{A_{13}^{-1}}{A_{33}^{-1}}=-\frac{d_5-d_6\nu_{23}}{2d_1}<0,$$
$$\nu_{32}=-\frac{A_{32}^{-1}}{A_{22}^{-1}}=\frac{d_2\nu_{23}}{d_3+d_2\nu_{23}^2}<0,$$
$$\nu_{31}=-\frac{A_{31}^{-1}}{A_{11}^{-1}}=\frac{4d_1^2d_2\nu_{13}}{4d_1d_2d_3+d_3d_6^2+4d_1^2d_2\nu_{13}^2}<0,$$
$$\nu_{12}=-\frac{A_{12}^{-1}}{A_{22}^{-1}}=-\frac{d_3d_6+2d_1d_2\nu_{23}\nu_{13}}{2d_1\left(d_3+d_2\nu_{23}^2\right)}<0,$$
$$\nu_{21}=-\frac{A_{21}^{-1}}{A_{11}^{-1}}=\frac{4d_1^2\left(d_3+d_2\nu_{23}^2\right)\nu_{12}}{4d_1d_2d_3+d_3d_6^2+4d_1^2d_2\nu_{13}^2}<0.$$

Although there are relatively few materials known to have negative Poisson's ratios, the above examples of such materials — for which the equations for each displacement u_i are independent of one another — can serve as a basis for the development of composite materials with similar properties. In accordance with the classification in [12], materials with negative Poisson's ratios are naturally hard substances — as opposed to materials which are more like fluids and have positive Poisson's ratios.

Monoclinic syngony (axis of symmetry of the second order x_1)

$$\mu_{5,6}=\frac{1}{2}\left[\frac{1}{2}(A_{55}+A_{66})+A_{31}+A_{21}\pm\sqrt{\left(\frac{1}{2}(A_{55}-A_{66})+A_{31}-A_{21}\right)^2+2A_{41}^2}\right]=$$
$$=\frac{1}{2}d_1\left[\frac{d_5+d_6}{2d_1}+c_{31}+c_{21}\pm\sqrt{\left(\frac{d_5-d_6}{2d_1}+c_{31}-c_{21}\right)^2+2c_{41}^2}\right],$$

$$f_{ip}=\begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} & 0 & 0\\ f_{21} & f_{22} & f_{23} & f_{24} & 0 & 0\\ f_{31} & f_{32} & f_{33} & f_{34} & 0 & 0\\ f_{41} & f_{42} & f_{43} & f_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & \cos\alpha & -\sin\alpha\\ 0 & 0 & 0 & 0 & \sin\alpha & \cos\alpha \end{bmatrix}, \quad f_{ip}f_{iq}=\delta_{pq},$$

$$\operatorname{tg}2\alpha=\frac{\sqrt{2}\,A_{41}}{\frac{1}{2}(A_{55}-A_{66})+A_{31}-A_{21}}=\frac{\sqrt{2}\,c_{41}}{\frac{d_5-d_6}{2d_1}+c_{31}-c_{21}}.$$

The eigenvalues μ_1, μ_2, μ_3, and μ_4 are roots of the fourth-degree equation

$$\begin{vmatrix} A_{11}-\mu & & & \\ \frac{1}{2}A_{66} & A_{22}-\mu & \text{sym} & \\ \frac{1}{2}A_{55} & \frac{1}{2}A_{44} & A_{33}-\mu & \\ 0 & A_{42} & A_{43} & \frac{1}{2}A_{44}+A_{32}-\mu \end{vmatrix}=0.$$

$$A_{11}=d_1,\ A_{22}=d_1c_{21}^2+d_2,\ A_{33}=d_1c_{31}^2+d_2c_{32}^2+d_3,$$
$$A_{44}=d_1c_{41}^2+d_2c_{42}^2+d_3c_{43}^2+d_4,\ A_{32}=d_1c_{31}c_{21}+d_2c_{32},$$
$$A_{55}=d_5,\ A_{66}=d_6,\ A_{42}=d_1c_{41}c_{21}+d_2c_{42},$$
$$A_{43}=d_1c_{41}c_{31}+d_2c_{42}c_{32}+d_3c_{43}$$

and depend on arbitrary parameters $d_i > 0$, c_{ip}. In this case, there are the inequalities

$$0\leqslant\mu_6\leqslant\mu_5,\quad\text{if}\quad \frac{d_5+d_6}{2d_1}+c_{31}+c_{21}\geqslant 0,\ \left(\frac{d_5}{2d_1}+c_{31}\right)\left(\frac{d_6}{2d_1}+c_{21}\right)\geqslant\frac{1}{2}c_{41}^2;$$
$$\mu_6\leqslant 0\leqslant\mu_5,\quad\text{if}\quad \left(\frac{d_5}{2d_1}+c_{31}\right)\left(\frac{d_6}{2d_1}+c_{21}\right)\leqslant\frac{1}{2}c_{41}^2;$$
$$\mu_6\leqslant\mu_5\leqslant 0,\quad\text{if}\quad \frac{d_5+d_6}{2d_1}+c_{31}+c_{21}\leqslant 0,\ \left(\frac{d_5}{2d_1}+c_{31}\right)\left(\frac{d_6}{2d_1}+c_{21}\right)\geqslant\frac{1}{2}c_{41}^2.$$

If necessary, the unwritten eigenvalues μ_i for materials with rhombic and monoclinic syngony can be expressed through A_{ij} in accordance with formulas for the roots of third- and fourth-degree polynomials [13]. The eigenvectors f_{ip} are then found.

Triclinic Syngony. In this case, the matrix A^*_{ik} has the general form of (5-6). To avoid having to solve a sixth-degree characteristic equation for the matrix (6), we can assign μ_i and f_{ip} arbitrarily in Eqs. (5). However, in this case it is necessary to ensure satisfaction of the conditions of positive-definiteness [9] for A_{ij}.

Taking (4-5) into account, we write Eqs. (1) in the form

$$(\mu_1 f_{ij11}\tilde\partial_{11}+\mu_2 f_{ij22}\tilde\partial_{22}+\mu_3 f_{ij33}\tilde\partial_{33}+\mu_4 2f_{ij23}\tilde\partial_{23}+\mu_5 2f_{ij31}\tilde\partial_{31}+$$
$$+\mu_6 2f_{ij12}\tilde\partial_{12}-\rho\delta_{ij}\partial_{..})u_j+F_i=0. \tag{15}$$

where $\tilde{\partial}_{rs} = \tilde{\partial}_{sr} = f_{k\ell rs}\partial_{k\ell}$ are differential operators which are invariant under an orthogonal coordinate transformation, since they are the convolution of two symmetric second-rank tensors. Due to the orthogonality of the characteristic tensors, we also have $\partial_{k\ell} = f_{k\ell rs}\tilde{\partial}_{rs}$.

In Eqs. (15), the parameters $\mu_1, \ldots, \mu_6$ are invariant as eigenvalues of the symmetric matrix A^*_{ik}. Numbering μ_i in decreasing order ($\mu_1 \geq \mu_2 \geq \ldots \geq \mu_6$), we can classify Eqs. (15) (anisotropic materials) in relation to the number of different eigenvalues μ_k and their multiplicity. A similar classification of anisotropic materials in relation to the number of different characteristic moduli λ_k of matrix A_{ij} and their multiplicities was presented in [6, 8].

We place each Eq. (15) into correspondence with the symbols $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}^*$, where $k \leq 6$, $\alpha_k \geq 1$, $\alpha_1 + \alpha_2 + \ldots + \alpha_k = 6$. Here, k is the number of different characteristic parameters μ_i; α_i is their multiplicity. Equations (15) break down into 32 classes [6, 8], each class corresponding to a certain symbol $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}^*$:

1) $\{6\}^*$;
2) $\{1, 5\}^*$, $\{2, 4\}^*$, $\{3, 3\}^*$, $\{4, 2\}^*$, $\{5, 1\}^*$;
3) $\{1, 1, 4\}^*$, $\{1, 2, 3\}^*$, $\{1, 3, 2\}^*$, $\{1, 4, 1\}^*$, $\{2, 1, 3\}^*$, $\{2, 2, 2\}^*$, $\{2, 3, 1\}^*$, $\{3, 1, 2\}^*$, $\{3, 2, 1\}^*$, $\{4, 1, 1\}^*$;
4) $\{1, 1, 1, 3\}^*$, $\{1, 1, 2, 2\}^*$, $\{1, 1, 3, 1\}^*$, $\{1, 2, 1, 2\}^*$, $\{1, 2, 2, 1\}^*$, $\{1, 3, 1, 1\}^*$, $\{2, 1, 1, 2\}^*$, $\{2, 1, 2, 1\}^*$, $\{2, 2, 1, 1\}^*$, $\{3, 1, 1, 1\}^*$;
5) $\{1, 1, 1, 1, 2\}^*$, $\{1, 1, 1, 2, 1\}^*$, $\{1, 1, 2, 1, 1\}^*$, $\{1, 2, 1, 1, 1\}^*$, $\{2, 1, 1, 1, 1\}^*$;
6) $\{1, 1, 1, 1, 1, 1\}^*$.

The order of the numbers within the given symbol is important. Equations of a different class are obtained when nonidentical numbers in the symbol are rearranged. A more detailed classification of Eqs. (15) should be made in relation to the form of the characteristic tensor f_{ijpq}. In [5, 6], the characteristic tensors f_{ijpq} were constructed in general form in terms of 15 arbitrary parameters.

Not all 32 classes of equations are possible for physically existing materials; only those for which positive-definiteness of matrix (7) is assured are possible [9]. In particular, class $\{6\}^*$ is impossible. A separate study of the above-found classes of equations should be made within the framework of the mathematical theory of elasticity.

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